A generalized quantum nonlinear oscillator

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Abstract

We examine various generalizations, e.g. exactly solvable, quasi-exactly solvable and non-Hermitian variants, of a quantum nonlinear oscillator. For all these cases, the same mass function has been used and it has also been shown that the new exactly solvable potentials possess shape invariance symmetry. The solutions are obtained in terms of classical orthogonal polynomials.

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1. Introduction

Recently, there has been a surge of interest in obtaining exact [1] and quasi-exact solutions [2] to the position-dependent mass Schrödinger equation (PDMSE) for various potentials and mass functions by using various methods such as Lie algebraic techniques [3], supersymmetric quantum mechanics (factorization method) [4, 5], the shape invariance approach [6], point canonical transformation [7], path integral formalism [8], the transfer matrix method [9], etc. Apart from the intrinsic interest, the motivation behind this issue arises because of the relevance of position-dependent mass in describing the physics of many microstructures of current interest, such as compositionally graded crystals [10], quantum dots [11], ^{3}He clusters [12], metal clusters [13], etc. The concept of position-dependent mass comes from the effective mass approximation [14] which is a useful tool for studying the motion of carrier electrons in pure crystals and also for the virtual-crystal approximation in the treatment of homogeneous alloys (where the actual potential is approximated by a periodic potential) as well as in graded mixed semiconductors (where the potential is not periodic). Attention to the effective mass approach stems from the extraordinary development in crystallographic growth techniques, which allow for the production of a non-uniform semiconductor specimen with abrupt heterojunctions. In these mesoscopic materials, the effective mass of the charge carriers is position dependent. Consequently, the study of the effective mass Schrödinger equation becomes relevant for a deeper understanding of the non-trivial quantum effects observed on these nanostructures. The position-dependent (effective) mass is also used in the construction of pseudo-potentials, which have a significant computational advantage in the quantum Monte

Carlo method [15]. It has also been found that such equations appear in very different areas. For example, it has been shown that a constant mass Schrödinger equation in curved space and equations based on deformed commutation relations can be interpreted in terms of PDMSE in the flat space [16] and a $\mathcal{PT}$ symmetric cubic anharmonic oscillator [17].

The nonlinear differential equation

$$(1 + \lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + \alpha^2 x = 0, \qquad \lambda > 0, \tag{1}$$

was studied by Mathews and Lakshmanan in [18, 19] as an example of a nonlinear oscillator, and it was shown that the solution of (1) is

$$x = A\sin(\omega t + \phi) \tag{2}$$

with the following additional restriction linking frequency and amplitude:

$$\omega^2 = \frac{\alpha^2}{1 + \lambda A^2}. \tag{3}$$

Furthermore, (1) can be obtained from the Lagrangian [18]

$$L = \frac{1}{2} \frac{1}{(1 + \lambda x^2)} (\dot{x}^2 - \alpha^2 x^2) \tag{4}$$

so that both the kinetic and the potential terms depend on the same parameter λ. So this nonlinear oscillator must be considered as a particular case of a system with a position-dependent effective mass. Recently in a series of papers [20, 21], this particular nonlinear system has been generalized to higher dimensions and various properties of this system have been studied. The classical Hamiltonian corresponding to the λ-dependent oscillator is given by [18, 21]

$$H = \left(\frac{1}{2m}\right) P_x^2 + \left(\frac{1}{2}\right) g \left(\frac{x^2}{1 + \lambda x^2}\right), \qquad P_x = \sqrt{1 + \lambda x^2}\, p_x, \qquad g = m\alpha^2, \tag{5}$$

p_x being the canonically conjugate momentum defined by $p_x = \frac{\partial L}{\partial \dot{x}}$, L the Lagrangian and m the mass.

It has been shown in [21] that in the space $\mathcal{L}^2(\Re, d\mu)$ where $d\mu = \left(\frac{1}{\sqrt{1 + \lambda x^2}}\right) dx$, the differential operator $\sqrt{1 + \lambda x^2}\frac{d}{dx}$ is skew self-adjoint. Therefore, in contrast to the naive expectation of ordering ambiguities, the transition from the classical system to the quantum one is given by defining the momentum operator

$$P_x = -i\sqrt{1 + \lambda x^2}\frac{d}{dx} \tag{6}$$

so that

$$(1 + \lambda x^2)p_x^2 \rightarrow -\left(\sqrt{1 + \lambda x^2}\frac{d}{dx}\right)\left(\sqrt{1 + \lambda x^2}\frac{d}{dx}\right).$$

Therefore, the quantum version of the Hamiltonian (5) with $\hbar = 1$ becomes [21]

$$\hat{H} = -\frac{1}{2m}(1 + \lambda x^2)\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)\lambda x \frac{d}{dx} + \frac{1}{2}g\left(\frac{1}{1 + \lambda x^2}\right), \tag{7}$$

where $g = \alpha(m\alpha + \lambda)$. It is to be noted that in [21], the value of the parameter g has been slightly modified from that given in equation (5).

It may be pointed out that this λ-dependent system can be considered as a deformation of the standard harmonic oscillator in the sense that for $\lambda \rightarrow 0$, all the characteristics of the linear oscillator are recovered.

In [21], PDMSE corresponding to this nonlinear oscillator has been solved exactly as a Sturm–Liouville problem, and λ-dependent eigenvalues and eigenfunctions were obtained for

both $\lambda > 0$ and $\lambda < 0$. The λ-dependent wavefunctions were shown to be related to a family of orthogonal polynomials that can be considered as λ-deformations of the standard Hermite polynomials. Also, the Schrödinger factorization formalism, intertwining method and shape invariance approach were discussed with reference to this particular quantum Hamiltonian. The existence of a λ-dependent Rodrigues formula, a generating function and λ-dependent recursion relations were obtained.

In this paper our objective is to re-examine this problem and obtain a closed-form expression for the normalization constant, modified generating function and recursion relations for $\Lambda\left(= \frac{\lambda}{\alpha}\right)$-deformed Hermite polynomials. A relation between the Λ-deformed Hermite polynomials and Jacobi polynomials will also be obtained. We shall also obtain a number of exactly solvable, quasi-exactly solvable and non-Hermitian potentials corresponding to the same mass function $m(x) = (1 + \lambda x^2)^{-1}$. It will be seen that some of these potentials are generalizations of the nonlinear oscillator potentials while the others are of different types. It will be shown that these exactly solvable potentials are shape invariant. Moreover, these potentials can also be complexified and by doing so we shall also obtain a number of exactly solvable non-Hermitian potentials within the framework of PDMSE. As a method of obtaining these results, we shall use point canonical transformation consisting of a change of coordinate only. The organization of the paper is as follows: in section 2, we shall obtain exactly solvable potentials and a relation between Λ-deformed Hermite polynomials and Jacobi polynomials; in section 3, it is shown that the exactly solvable potentials are shape invariant; in section 4, we obtain exactly solvable non-Hermitian potentials; section 5 deals with complex quasi-exactly solvable potentials and finally section 6 is devoted to a discussion.

2. Exactly solvable potentials for the mass $m(x) = \left(\frac{1}{1+\lambda x^2}\right)$

Here we shall obtain exact solutions to PDMSE for a number of potentials with the same mass function $m(x) = \left(\frac{1}{1+\lambda x^2}\right)$. For this purpose, we first write PDMSE corresponding to the Hamiltonian given in equation (7) with $m = 1$ and $\lambda > 0$ as

$$\left[-(1 + \lambda x^2)\frac{d^2\psi}{dx^2} - \lambda x \frac{d\psi}{dx} - \frac{g}{\lambda}\left(\frac{1}{1 + \lambda x^2}\right)\right]\psi = E\psi \tag{8}$$

$$E = 2e - \frac{g}{\lambda}, \tag{9}$$

where e is the energy for the Hamiltonian (7). Now expanding $(1 + \lambda x^2)^{-1}$ for $|x| < \frac{1}{\sqrt{\lambda}}$, we can write the potential of equation (8) as

$$V(x) = -\frac{g}{\lambda} + gx^2 - \lambda O(x^3). \tag{10}$$

It is clear from (10) that the term $\left(-\frac{g}{\lambda}\right)$ in equation (9) cancels from both sides of equation (8), so that the new eigenvalues (9) are actually the old eigenvalues e of the Hamiltonian (7). Also, as $\lambda \to 0$, the potential and the eigenvalues of equation (8) reduce to those of a linear harmonic oscillator.

Now generalizing the potential of equation (8) as below, the corresponding PDMSE reads as

$$-(1 + \lambda x^2)\frac{d^2\psi}{dx^2} - \lambda x \frac{d\psi}{dx} + \left[\frac{B^2 - A^2 - A\sqrt{\lambda}}{1 + \lambda x^2} + B(2A + \sqrt{\lambda})\left(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}\right) + A^2\right]\psi = E\psi. \tag{11}$$

It is seen from (11) that if we put $B = 0$, then the potential reduces to that of the nonlinear oscillator with $\frac{g}{\lambda} = A^2 + A\sqrt{\lambda}$. It is to be noted that this generalization should correctly reproduce the $\lambda \to 0$ limit, in which case equation (11) reduces to the Schrödinger equation for a linear harmonic oscillator. In the appendix we have shown that in the limit $\lambda \to 0$ and for $A = \frac{\alpha}{\sqrt{\lambda}}$ (which is one of the solution of the quadratic equation $A^2 + A\sqrt{\lambda} = \frac{g}{\lambda}$), $B = 0$ the potential of equation (11), the energy eigenvalues (18) and the wavefunction given in (19) reduce to those of a linear harmonic oscillator. This particular generalization is made so that it corresponds to the hyperbolic Scarf II potential [22] in the constant mass case. In order to solve (11), we now perform a transformation involving a change of variable given by

$$z = \int \frac{\mathrm{d}x}{\sqrt{F(x)}} = \frac{1}{\sqrt{\lambda}} \sinh^{-1}(\sqrt{\lambda}x), \tag{12}$$

where

$$F(x) = 1 + \lambda x^2, \qquad \lambda > 0. \tag{13}$$

Under transformation (12), equation (11) reduces to a Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + V(z)\psi(z) = E\psi(z), \tag{14}$$

where the potential $V(z)$ is given by

$$V(z) = (B^2 - A^2 - A\sqrt{\lambda}) \operatorname{sech}^2(z\sqrt{\lambda}) + B(2A + \sqrt{\lambda}) \tanh(z\sqrt{\lambda}) \operatorname{sech}(z\sqrt{\lambda}) + A^2. \tag{15}$$

Potential (15) is a standard solvable potential and the solutions are given by [22]

$$\psi_n(z) = N_n i^n (1 + \sinh^2(z\sqrt{\lambda}))^{-\frac{s}{2}} \mathrm{e}^{-r\tan^{-1}(\sinh(z\sqrt{\lambda}))} P_n^{(ir-s-\frac{1}{2}, -ir-s-\frac{1}{2})}(i \sinh(z\sqrt{\lambda}), \tag{16}$$

where N_n is the normalization constant, $s = \frac{A}{\sqrt{\lambda}}$, $r = \frac{B}{\sqrt{\lambda}}$ and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [24]. The normalization constants $N_n, n = 0, 1, 2, \ldots$, are given by [23]

$$N_n = \left[\frac{\sqrt{\lambda} n! (s - n) \Gamma\left(s - ir - n + \frac{1}{2}\right) \Gamma\left(s + ir - n + \frac{1}{2}\right)}{\pi 2^{-2s} \Gamma(2s - n + 1)} \right]^{1/2}. \tag{17}$$

The eigenvalues E_n are given by

$$E_n = n\sqrt{\lambda}(2A - n\sqrt{\lambda}), \qquad n = 0, 1, 2, \ldots < s. \tag{18}$$

Subsequently by performing the inverse of transformation (12), we find the solution to PDMSE (11) as

$$\psi_n(x) = \left[\frac{\sqrt{\lambda} n! (s - n) \Gamma\left(s - ir - n + \frac{1}{2}\right) \Gamma\left(s + ir - n + \frac{1}{2}\right)}{\pi 2^{-2s} \Gamma(2s - n + 1)} \right]^{1/2}$$

$$i^n (1 + \lambda x^2)^{-\frac{s}{2}} \mathrm{e}^{-r\tan^{-1}(x\sqrt{\lambda})} P_n^{(ir-s-\frac{1}{2}, -ir-s-\frac{1}{2})}(ix\sqrt{\lambda}), \qquad n = 0, 1, 2, \cdots < s \left(= \frac{A}{\sqrt{\lambda}}\right). \tag{19}$$

At this point, it is natural to ask the following question: are there other solvable potentials corresponding to the mass function $m(x) = \left(\frac{1}{1+\lambda x^2}\right)$? The answer to this question is in the affirmative. The procedure to obtain these potentials is similar and so instead of treating each case separately, we have presented the potentials and the corresponding solutions in table 1. The first two and the last two potentials in table 1 are actually the generalizations of the nonlinear oscillator potential. Although the other two potentials in the table are not generalizations of the nonlinear oscillator potential, nevertheless they are exactly solvable potentials with the same mass function.

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Table 1. Exactly solvable shape invariant potentials $V(x)$, superpotential $W(x)$, energy eigenvalue E_n and wavefunctions $\psi_n(x)$, where $s = \frac{A}{\lambda}, r = \frac{B}{\sqrt{\lambda}}, r_1 = \frac{B}{\lambda}, a = \frac{r_1}{s-n}, s_1 = s - n + a, s_2 = s - n - a, s_3 = a - n - s, s_4 = -(s + n + a), s' = \frac{A}{\sqrt{|\lambda|}} \& r' = \frac{B}{\sqrt{|\lambda|}}$. The first four entries correspond to $\lambda > 0$ and the last two correspond to $\lambda < 0$.

$V(x)$	$W(x)$	E_n	$\psi_n(x)$	$a_i, i =$ $0, 1, \ldots$	$R(a_i)$
$\frac{B^2-A^2-A\sqrt{\lambda}}{1+\lambda x^2} + B(2A + \sqrt{\lambda})\frac{\sqrt{\lambda}x}{1+\lambda x^2} + A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}} + B\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{\lambda}(2A - n\sqrt{\lambda})$	$i^n(1+\lambda x^2)^{-\frac{s}{2}}e^{-r\tan^{-1}(x\sqrt{\lambda})}$ $P_n^{(ir-s-\frac{1}{2}, -ir-s-\frac{1}{2})}(ix\sqrt{\lambda})$	$(A - i\sqrt{\lambda}, B)$	$\sqrt{\lambda}[2A - (2i + 1)\sqrt{\lambda}]$
$A^2 + \frac{B^2}{A^2} - \frac{A(A+\sqrt{\lambda})}{1+\lambda x^2} + 2B\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}}, B < A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}} + \frac{B}{A}$	$A^2 + \frac{B^2}{A^2} - (A - n\sqrt{\lambda})^2 - \frac{B^2}{(A-n\sqrt{\lambda})^2}$	$\left(1 - \frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_1}{2}} \left(1 + \frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_2}{2}}$ $P_n^{(s_1,s_2)}\left(\frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)$	$(A - i\sqrt{\lambda}, B)$	$A^2 - [A - (i + 1)\sqrt{\lambda}]^2 + \frac{B^2}{A^2} - \frac{B^2}{[A-(i+1)\sqrt{\lambda}]^2}$
$A^2 + \frac{B^2}{A^2} - 2B\frac{\sqrt{1+\lambda x^2}}{\sqrt{\lambda}x} + \frac{A(A-\sqrt{\lambda})}{\lambda x^2}, B > A^2$ $0 \leqslant x\sqrt{\lambda} \leqslant \infty$	$\frac{B}{A} - A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}$	$A^2 + \frac{B^2}{A^2} - (A + n\sqrt{\lambda})^2 - \frac{B^2}{(A+n\sqrt{\lambda})^2}$	$\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} - 1\right)^{\frac{s_3}{2}} \left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} + 1\right)^{\frac{s_4}{2}}$ $P_n^{(s_3,s_4)}\left(\frac{1+\lambda x^2}{x\sqrt{\lambda}}\right)$	$(A + i\sqrt{\lambda}, B)$	$A^2 - [A + (i + 1)\sqrt{\lambda}]^2 + \frac{B^2}{A^2} - \frac{B^2}{[A+(i+1)\sqrt{\lambda}]^2}$
$\frac{A^2+B^2+A\sqrt{\lambda}}{\lambda x^2} - B(2A + \lambda)\frac{\sqrt{1+\lambda x^2}}{\lambda x^2} + A^2, A < B$ $0 \leqslant x\sqrt{\lambda} \leqslant \infty$	$A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} - B\frac{1}{x\sqrt{\lambda}}$	$n\sqrt{\lambda}(2A - n\sqrt{\lambda})$	$(\sqrt{1+\lambda x^2} - 1)^{(\frac{r-s}{2})}(\sqrt{1+\lambda x^2} + 1)^{-(\frac{r+s}{2})}$ $P_n^{(r-s-\frac{1}{2}, -r-s-\frac{1}{2})}(\sqrt{1+\lambda x^2})$	$(A - i\sqrt{\lambda}, B)$	$\sqrt{\lambda}[2A - (2i + 1)\sqrt{\lambda}]$
$\frac{A^2+B^2-A\sqrt{|\lambda|}}{1+\lambda x^2} - B(2A - \sqrt{|\lambda|})\frac{x\sqrt{|\lambda|}}{1+\lambda x^2} - A^2$ $\frac{1}{-\sqrt{|\lambda|}} \leqslant x \leqslant \frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}} - B\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{|\lambda|}(2A + n\sqrt{|\lambda|})$	$(1 - x\sqrt{|\lambda|})^{(\frac{s'-r'}{2})}(1 + x\sqrt{|\lambda|})^{(\frac{r'+s'}{2})}$ $P_n^{(s'-r'-\frac{1}{2}, s'+r'-\frac{1}{2})}\left(x\sqrt{|\lambda|}\right)$	$(A + i\sqrt{|\lambda|}, B)$	$\sqrt{|\lambda|}\left[2A + (2i + 1)\sqrt{|\lambda|}\right]$
$\frac{A(A-\sqrt{|\lambda|})}{1+\lambda x^2} - 2B\frac{x\sqrt{|\lambda|}}{1+\lambda x^2} - A^2 + \frac{B^2}{A^2}$ $\frac{1}{-\sqrt{|\lambda|}} \leqslant x \leqslant \frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}} - \frac{B}{A}$	$\frac{B^2}{A^2} - A^2 + (A + n\sqrt{|\lambda|})^2 - \frac{B^2}{(A+n\sqrt{|\lambda|})^2}$	$\left(\frac{\lambda x^2}{1+\lambda x^2} - 1\right)^{-(\frac{s'+n}{2})}e^{-a\sqrt{|\lambda|}x}$ $P_n^{(-s'-n-ia, -s'-n+ia)}\left(-i\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}}\right)$	$(A + i\sqrt{|\lambda|}, B)$	$\frac{B^2}{A^2} - \frac{B^2}{[A+(i+1)\sqrt{|\lambda|}]^2}$ $-A^2 + [A + (i + 1)\sqrt{|\lambda|}]^2$

2.1. Relation between the Λ-deformed Hermite polynomial and Jacobi polynomial, generating function, recursion relation

Here we shall obtain a correspondence between the Λ-deformed Hermite polynomials [21] and Jacobi polynomials. We recall that the Hamiltonian for the nonlinear oscillator is given by [21]

$$H = -\frac{1}{2}(1 + \lambda x^2)\frac{d^2}{dx^2} - \frac{1}{2}\lambda x\frac{d}{dx} + \frac{g}{2}\left(\frac{x^2}{1 + \lambda x^2}\right).$$

After introducing adimensional variables (y, Λ) as was done in [21]

$$y = \sqrt{\alpha}x, \qquad \Lambda = \frac{\lambda}{\alpha}, \tag{20}$$

the Schrödinger equation $H\psi = \epsilon\psi$ reduces to

$$\left[-\frac{1}{2}(1 + \Lambda y^2)\frac{d^2}{dy^2} - \frac{1}{2}\Lambda y\frac{d}{dy} + \frac{1 + \Lambda}{2}\left(\frac{y^2}{1 + \Lambda y^2}\right)\right]\psi = \epsilon\psi. \tag{21}$$

The eigenvalues and eigenfunctions for $\Lambda < 0$ are [21]

$$\psi_m(y, \Lambda) = \mathcal{H}_m(y, \Lambda)(1 - |\Lambda|y^2)^{\frac{1}{(2|\Lambda|)}}$$
$$\epsilon_m = \left(m + \tfrac{1}{2}\right) - \tfrac{1}{2}m^2\Lambda, \qquad m = 0, 1, 2, \ldots, \tag{22}$$

where $\mathcal{H}_m(y, \Lambda)$ is the Λ-deformed Hermite polynomial whose Rodrigues formula and generating function are given in (27). For $\Lambda > 0$,

$$\psi_m(y, \Lambda) = \mathcal{H}_m(y, \Lambda)(1 + \Lambda y^2)^{-\frac{1}{2\Lambda}}$$
$$\epsilon_m = \left(m + \tfrac{1}{2}\right) - \tfrac{1}{2}m^2\Lambda, \qquad m = 0, 1, 2\cdots, N_\Lambda, \tag{23}$$

where N_Λ denotes the greatest integer lower than $m_\Lambda\left(= \frac{1}{\Lambda}\right)$. On the other hand, putting $B = 0$ and $A = \frac{\alpha}{\sqrt{\lambda}}$ in solution (19) of equation (11), the eigenfunctions of equation (21) can be written in terms of the Jacobi polynomial as

$$\psi_n(y) = N_n(1 + \Lambda y^2)^{-\frac{1}{2\Lambda}} P_n^{(-\frac{1}{2} - \frac{1}{\Lambda}, -\frac{1}{2} - \frac{1}{\Lambda})}(iy\sqrt{\Lambda}), \quad n = 0, 1, 2\ldots < \frac{1}{\Lambda} \quad (\Lambda > 0). \tag{24}$$

For $\Lambda < 0$, putting $B = 0$, $A = \frac{\alpha}{\sqrt{|\lambda|}}$ in the wavefunction of the fifth entry of table 1 and using (20), we obtain

$$\psi_n(y) = N_n(1 + \Lambda y^2)^{-\frac{1}{2\Lambda}} P_n^{(-\frac{1}{2} - \frac{1}{\Lambda}, -\frac{1}{2} - \frac{1}{\Lambda})}(y\sqrt{|\Lambda|}), \quad n = 0, 1, 2\ldots \quad (\Lambda < 0). \tag{25}$$

Comparing equations (22) and (25) and also equations (23) and (24), it is possible to derive a relation between the Λ-deformed Hermite polynomial $\mathcal{H}_n(y, \Lambda)$ and the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ as

$$P_n^{(-\frac{1}{2} - \frac{1}{\Lambda}, -\frac{1}{2} - \frac{1}{\Lambda})}(iy\sqrt{\Lambda}) = \frac{1}{n!}\left(\frac{1}{2i\sqrt{\Lambda}}\right)^n \mathcal{H}_n(y, \Lambda), \quad \forall \Lambda. \tag{26}$$

The Rodrigues formula and the generating function for the Λ-deformed Hermite polynomial $\mathcal{H}_n(y, \Lambda)$ are given by [21]

$$\mathcal{H}_n(y, \Lambda) = (-1)^n z_y^{\frac{1}{\Lambda} + \frac{1}{2}} \frac{d^n}{dy^n}\left[z_y^n z_y^{-(\frac{1}{\Lambda} + \frac{1}{2})}\right], \qquad z_y = 1 + \Lambda y^2,$$
$$\mathcal{F}(t, y, \Lambda) = (1 + \Lambda(2ty - t^2))^{\frac{1}{\Lambda}}. \tag{27}$$

It was shown [21] that the polynomials obtained from the generating function $\mathcal{F}(t, y, \Lambda)$ with those obtained from the Rodrigues formula are essentially the same and only differ in

the values of the global multiplicative coefficients. We have observed that if the generating function $\mathcal{F}(t, y, \Lambda)$ is taken as

$$(1 + \Lambda(2ty - t^2))^{\frac{1}{\Lambda}} = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\left(-\frac{1}{\Lambda}\right)_n}{\left(\frac{1}{2} - \frac{1}{\Lambda}\right)_n} \mathcal{H}_n(y, \Lambda) \frac{t^n}{n!}, \tag{28}$$

where $(a)_n$ represents the Pöchhammer symbol given by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, then the polynomials obtained from the above relation are exactly similar to those obtained from the Rodrigues formula given in equation (27).

Correspondingly, the recursion relations are obtained as

$$(\Lambda(2n + 1) - 2)[2(1 - n\Lambda)y\mathcal{H}_n(y, \Lambda) + (\Lambda(2n - 1) - 2)n\mathcal{H}_{n-1}(y, \Lambda)]$$
$$= (n\Lambda - 2)\mathcal{H}_{n+1}(y, \Lambda) \tag{29}$$

and

$$(\Lambda(n - 2) - 2)[2(\Lambda(2n - 1) - 2)n\mathcal{H}_n(y, \Lambda) - (\Lambda(n - 1) - 2)\mathcal{H}'_n(y, \Lambda)]$$
$$= n\Lambda(\Lambda(2n - 1) - 2)[2(\Lambda(n - 2) - 2)y\mathcal{H}'_{n-1}(y, \Lambda)$$
$$- (n - 1)(\Lambda(2n - 3) - 2)\mathcal{H}'_{n-2}(y, \Lambda)], \tag{30}$$

where the 'prime' denotes differentiation with respect to y. For $\Lambda \to 0$, equations (29) and (30) give the recursion relations for the Hermite polynomial [24].

3. Shape invariance approach to supersymmetric PDMSE

The supersymmetric approach to PDMSE [5] may be discussed either by reducing PDMSE to a constant mass Schrödinger equation or by starting with modified intertwining operators consisting of first-order differential operators. Here, we shall be following the latter approach. Thus, we consider operators of the form

$$A = P_x - iW(x), \qquad A^\dagger = P_x + iW(x), \qquad P_x = \frac{1}{\sqrt{m(x)}}\left(-i\frac{d}{dx}\right). \tag{31}$$

We now consider the supercharges $Q, Q^\dagger$ defined by

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \tag{32}$$

The supersymmetric Hamiltonian is then obtained as

$$H^{\text{PDM}} = \{Q, Q^\dagger\} = \begin{pmatrix} H^{\text{PDM}}_- & 0 \\ 0 & H^{\text{PDM}}_+ \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}, \tag{33}$$

where the component Hamiltonians are given by

$$H^{\text{PDM}}_\pm = -\frac{1}{m(x)}\frac{d^2}{dx^2} + \left(\frac{m'}{2m^2}\right)\frac{d}{dx} + W^2 \pm \frac{W'}{\sqrt{m}}. \tag{34}$$

The Hamiltonians $H^{\text{PDM}}_\pm$ are supersymmetric partners and the potentials are

$$V^{\text{PDM}}_\pm = W^2(x) \pm \frac{W'(x)}{\sqrt{m(x)}}. \tag{35}$$

It can be easily seen that the following commutation and anticommutation relations

$$Q^2 = Q^{\dagger 2} = [Q, H^{\text{PDM}}] = [Q^\dagger, H^{\text{PDM}}] = 0$$
$$\{Q, Q^\dagger\} = \{Q^\dagger, Q\} = 0 \tag{36}$$

together with equation (33) complete the standard supersymmetry algebra [22, 25]. For unbroken supersymmetry (SUSY), the ground state of H_- has zero energy $\left(E_0^{(-)} = 0\right)$ provided the ground-state wavefunction $\psi_0^{(-)}(z)$ given by $\left(A\psi_0^{(-)} = 0\right)$

$$\psi_0^{(-)}(x) = N_0 \exp\left[-\int^x \sqrt{m(y)}W(y)\,\mathrm{d}y\right] \tag{37}$$

is normalizable. In this case it can be shown that, apart from the ground state of H_-, the partner Hamiltonians $H_\pm$ have identical bound-state spectra. In particular, they satisfy

$$E_{n+1}^{(-)} = E_n^{(+)}, \qquad n = 0, 1, 2 \ldots. \tag{38}$$

The eigenfunctions of $H_\pm$ corresponding to the same eigenvalue are related by

$$\begin{aligned} A\psi_{n+1}^{(-)} &= \left(E_n^{(+)}\right)^{\frac{1}{2}} \psi_n^{(+)}(x) \\ A^\dagger\psi_n^{(+)}(x) &= \left(E_n^{(+)}\right)^{\frac{1}{2}} \psi_{n+1}^{(-)}(x). \end{aligned} \tag{39}$$

It may be noted here that the superpotential $W(x)$ and therefore the factorization of the Hamiltonian could be generated from the ground-state solution of the equation. In a remarkable paper [26], Gendenshtein explored the relationship between SUSY and solvable potentials. The pair of potentials $V_\pm(x, a_0)$, a_0 being a set of parameters, is called shape invariant if it satisfies the relationship [5, 22]

$$\begin{aligned} V_+(x, a_0) &= W^2(x, a_0) + W'(x, a_0) \\ &= W^2(x, a_1) - W'(x, a_1) + R(a_0) \\ &= V(x, a_1) + R(a_0), \end{aligned} \tag{40}$$

where a_1 is some function of a_0 and $R(a_0)$ is independent of x. When SUSY is unbroken, the energy spectrum of any shape-invariant potential is given by [22]

$$E_n^{(-)} = \sum_{i=0}^{n-1} R(a_i), \qquad E_0^{(-)} = 0. \tag{41}$$

We are now going to study the factorization and the shape invariance property of the potentials for PDMSE. As an example, let us consider the generalized nonlinear oscillator of section 2. For this, it is now necessary to choose the superpotential $W(x)$ so that H_- can be identified with the Hamiltonian of equation (11). In this case, we choose the superpotential to be

$$W = A\frac{\sqrt{\lambda}x}{\sqrt{1 + \lambda x^2}} + B\frac{1}{\sqrt{1 + \lambda x^2}}. \tag{42}$$

Therefore, the Hamiltonians H_-^{PDM} and H_+^{PDM} can be factorized as

$$\begin{aligned} H_-^{\mathrm{PDM}} &= A^\dagger A \\ &= -(1 + \lambda x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{B^2 - A^2 - A\sqrt{\lambda}}{1 + \lambda x^2} + B(2A + \sqrt{\lambda})\left(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}\right) + A^2 \\ H_+^{\mathrm{PDM}} &= AA^\dagger \\ &= -(1 + \lambda x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{B^2 - A^2 + A\sqrt{\lambda}}{1 + \lambda x^2} + B(2A - \sqrt{\lambda})\left(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}\right) + A^2. \end{aligned} \tag{43}$$

These two Hamiltonians are related by

$$H_+^{\mathrm{PDM}}(x; A, B) = H_-^{\mathrm{PDM}}(x; A - \sqrt{\lambda}, B) + \sqrt{\lambda}(2A - \sqrt{\lambda}) \tag{44}$$

so that they satisfy the shape invariance condition

$$H_+^{\mathrm{PDM}}(x, a_0) = H_-^{\mathrm{PDM}}(x, a_1) + R(a_0), \tag{45}$$

where $\{a_0\} = (A, B)$, $\{a_1\} = (A - \sqrt{\lambda}, B)$ and $R(a_0) = \sqrt{\lambda}(2A - \sqrt{\lambda})$.

The ground state $\psi_0(x, a_0)$ of the Hamiltonian H_-^{PDM} is found by solving $A\psi_0(x, a_0) = 0$ and has a zero energy, i.e.

$$H_-^{\text{PDM}}(x, a_0)\psi_0(x, a_0) = 0. \tag{46}$$

Now using (45) we can see that $\psi_0(x, a_1)$ is an eigenstate of H_+^{PDM} with the energy $E_1 = R(a_0)$, because

$$
\begin{aligned}
H_+^{\text{PDM}}(x, a_0)\psi_0(x, a_1) &= H_-^{\text{PDM}}(x, a_1)\psi_0(x, a_1) + R(a_0)\psi_0(x, a_1) \\
&= R(a_0)\psi_0(x, a_1) \qquad \text{(using (46))}.
\end{aligned} \tag{47}
$$

Next, using the intertwining relation $H_-^{\text{PDM}}(x, a_0)A^\dagger(x, a_0) = A^\dagger(x, a_0)H_+^{\text{PDM}}(x, a_0)$ and equation (45), we see that

$$
\begin{aligned}
H_-^{\text{PDM}}(x, a_0)A^\dagger(x, a_0)\psi_0(x, a_1) &= A^\dagger(x, a_0)H_+^{\text{PDM}}(x, a_0)\psi_0(x, a_1) \\
&= A^\dagger[H_-^{\text{PDM}}(x, a_1) + R(a_0)]\psi_0(x, a_1)
\end{aligned} \tag{48}
$$

and hence using (46), we arrive at

$$H_-^{\text{PDM}}(x, a_0)A^\dagger(x, a_0)\psi_0(x, a_1) = R(a_1)A^\dagger(x, a_0)\psi_0(x, a_1). \tag{49}$$

This indicates that $A^\dagger(x, a_0)\psi_0(x, a_1)$ is an eigenstate of H_-^{PDM} with an energy $E_1 = R(a_0)$. Now iterating this process, we will find the sequence of energies for H_-^{PDM} as

$$E_n^{(-)} = \sum_{i=0}^{n-1} R(a_i) = n\sqrt{\lambda}(2A - n\sqrt{\lambda}), \qquad E_0^{(-)} = 0, \tag{50}$$

with corresponding eigenfunctions being

$$\psi_n(x, a_0) = A^\dagger(x, a_0)A^\dagger(x, a_1)\dots A^\dagger(x, a_{n-1})\psi_0(x, a_n), \tag{51}$$

where

$$a_i = f(a_{i-1}) = \underbrace{f(f(\dots(f(a_0))))}_{i \text{ times}} = (A - \mathrm{i}\sqrt{\lambda}, B) \quad \text{and} \quad R(a_i) = \sqrt{\lambda}[2(A - \mathrm{i}\sqrt{\lambda}) - \sqrt{\lambda}].$$

We have found a number of other potentials which are shape invariant for the same mass function. For all these potentials, the energy, wavefunctions and other parameters related to the shape invariance property are given in table 1.

3.1. Shape invariance approach to PDMSE with broken supersymmetry

When supersymmetry is broken, neither of the wavefunctions $\psi_0^{(\pm)}(x) \approx \exp[\pm \int^x \sqrt{m(y)}W(y)\,\mathrm{d}y]$ is normalizable and in this case all the energy values are degenerate, i.e. H_+ and H_- have identical energy eigenvalues [22, 26]

$$E_n^{(-)} = E_n^{(+)} \tag{52}$$

with ground-state energies greater than zero. So far as we know, little attention has been paid till now to study problems involving broken SUSY in the case of PDMSE. Broken supersymmetric shape invariant systems in the case of a constant mass Schrödinger equation have been discussed in [27]. Below, we illustrate the two-step procedure discussed in [28] for obtaining the energy spectra in PDMSE when the SUSY is broken. For this, we consider the superpotential as

$$W(x, A, B) = A\sqrt{|\lambda|}\frac{x}{\sqrt{1 + \lambda x^2}} - \frac{B}{\sqrt{|\lambda|}}\frac{\sqrt{1 + \lambda x^2}}{x}, \quad 0 < x < \frac{1}{\sqrt{|\lambda|}}, \quad \lambda < 0. \tag{53}$$

Then the supersymmetric partner potentials are obtained using (35) as

$$V_-(x, A, B) = \frac{A(A - \sqrt{|\lambda|})}{1 + \lambda x^2} - \frac{B(B - \sqrt{|\lambda|})}{\lambda x^2} - (A + B)^2$$
$$V_+(x, A, B) = \frac{A(A + \sqrt{|\lambda|})}{1 + \lambda x^2} - \frac{B(B + \sqrt{|\lambda|})}{\lambda x^2} - (A + B)^2. \tag{54}$$

The ground-state wavefunction is obtained from (37) as

$$\psi_0^{(-)} \sim x^{\frac{B}{\sqrt{|\lambda|}}} (1 + \lambda x^2)^{\frac{A}{2\sqrt{|\lambda|}}}. \tag{55}$$

For $A > 0$, $B > 0$ the ground-state wavefunction $\psi_0^{(-)}$ is normalizable which means that the SUSY is unbroken. But for $A > 0$, $B < 0$ and $A < 0$, $B > 0$, neither of $\psi_0^{(\pm)}$ is normalizable. Hence, SUSY is broken in both cases.

We shall discuss the case $A > 0$, $B < 0$. In this case, the eigenstates of $V_\pm(x, A, B)$ are related by

$$\psi_n^{(+)}(x, a_0) = A(x, a_0)\psi_n^{(-)}(x, a_0)$$
$$\psi_n^{(-)}(x, a_0) = A^\dagger(x, a_0)\psi_n^{(+)}(x, a_0), \tag{56}$$
$$E_n^{(-)}(a_0) = E_n^{(+)}(a_0).$$

Now we can show that the potentials in equation (54) are shape invariant by two different relations between the parameters.

Step 1.　The potentials of equation (54) are shape invariant if we change $A \to A + \sqrt{|\lambda|}$ and $B \to B + \sqrt{|\lambda|}$. The shape invariant condition is given by

$$V_+(x, A, B) = V_-(x, A + \sqrt{|\lambda|}, B + \sqrt{|\lambda|}) + (A + B + 2\sqrt{|\lambda|})^2 - (A + B)^2. \tag{57}$$

Now for $B < -\frac{1}{\sqrt{|\lambda|}}$, it is seen that the superpotential (53) resulting from the change of parameters as above falls in the class of a broken SUSY problem for which $E_0^{(-)} \neq 0$. Though the potentials of equation (54) are shape invariant but we are unable to determine the spectra for these potentials because of the absence of a zero energy ground state.

Another way of parameterizations $A \to A + \sqrt{|\lambda|}$ and $B \to -B$ gives us

$$V_+(x, A, B) = V_-(x, A + \sqrt{|\lambda|}, -B) + (A - B + \sqrt{|\lambda|})^2 - (A + B)^2, \tag{58}$$

which shows that V_- and V_+ are shape invariant. This change of parameters ($A \to A + \sqrt{|\lambda|}$ and $B \to -B$) leads to a system with unbroken SUSY since the parameter B changes sign. Hence, the ground-state energy of the potential $V_-(x, A + \sqrt{|\lambda|}, -B)$ is zero. From relation (58), we observe that $V_+(x, A, B)$ and $V_-(A + \sqrt{|\lambda|}, -B)$ differ only by a constant; hence, we have

$$\psi_+(x, A, B) = \psi_-(x, A + \sqrt{|\lambda|}, -B)$$
$$E_n^{(+)}(A, B) = E_n^{(-)}(x, A + \sqrt{|\lambda|}, -B) + (A - B + \sqrt{|\lambda|})^2 - (A + B)^2. \tag{59}$$

Thus, if we can evaluate the spectrum and energy eigenfunctions of unbroken SUSY $H_-^{\text{PDM}}(x, A + \sqrt{|\lambda|}, -B)$, then we can determine the spectrum and eigenfunctions $H_+^{\text{PDM}}(x, A, B)$ with broken SUSY. In the second step, we will do this.

Step 2. With the help of shape invariant formalism in the case of unbroken SUSY for PDMSE (see section 3), we obtain a spectrum and eigenfunctions for $V_-(x, A + \sqrt{|\lambda|}, -B)$ as

$$E_n^{(-)}(A + \sqrt{|\lambda|}, -B) = (A - B + \sqrt{|\lambda|} + 2n\sqrt{|\lambda|})^2 - (A - B + \sqrt{|\lambda|})^2$$
$$\psi_n^{(-)}(x, A + \sqrt{|\lambda|}, -B) \propto x^{\frac{B}{\sqrt{|\lambda|}}} (1 + \lambda x^2)^{\frac{A}{2\sqrt{|\lambda|}}} P_n^{(\frac{B}{\sqrt{|\lambda|}} - \frac{1}{2}, \frac{A}{\sqrt{|\lambda|}} - \frac{1}{2})} (1 + 2\lambda x^2). \tag{60}$$

Now using (60), (59) and (56), we obtain a spectrum and eigenfunctions for $V^-(x, A, B)$ with broken SUSY as

$$E_n^{(-)}(A, B) = (A - B + \sqrt{|\lambda|} + 2n\sqrt{|\lambda|})^2 - (A + B)^2$$
$$\psi_n^{(-)}(x, A, B) \propto x^{\frac{1-B}{\sqrt{|\lambda|}}}(1 + \lambda x^2)^{\frac{A}{2\sqrt{|\lambda|}}} P_n^{(\frac{1}{2} - \frac{B}{\sqrt{|\lambda|}}, \frac{A}{\sqrt{|\lambda|}} - \frac{1}{2})}(1 + 2\lambda x^2). \tag{61}$$

A similar approach can be applied in the case of $A < 0$ and $B > 0$. In this case, we change (A, B) into $(-A, B + \sqrt{|\lambda|})$ and the shape invariance condition is

$$V_+(x, A, B) = V_-(x, -A, B + \sqrt{|\lambda|}) + (B - A + \sqrt{|\lambda|})^2 - (A + B)^2, \tag{62}$$

and

$$E_n^{(-)}(A, B) = (B - A + \sqrt{|\lambda|} + 2n\sqrt{|\lambda|})^2 - (A + B)^2$$
$$\psi_n^{(-)}(x, A, B) \propto x^{\frac{1-A}{\sqrt{|\lambda|}}}(1 + \lambda x^2)^{\frac{B}{2\sqrt{|\lambda|}}} P_n^{(\frac{B}{\sqrt{|\lambda|}} - \frac{1}{2}, \frac{A}{\sqrt{|\lambda|}} - \frac{1}{2})}(1 + 2\lambda x^2). \tag{63}$$

4. Exactly solvable $\mathcal{PT}$ symmetric potentials in PDMSE

Here we shall find exactly solvable complex potentials, some of which are related to the nonlinear oscillator potential, within the framework of PDMSE. Before we consider any particular potential, let us note that a quantum mechanical Hamiltonian H is said to be $\mathcal{PT}$ symmetric [28] if

$$\mathcal{PT}\mathcal{H} = \mathcal{H}\mathcal{PT}, \tag{64}$$

where $\mathcal{P}$ is the parity operator acting as spatial reflection and $\mathcal{T}$ stands for time reversal, acting as the complex conjugation operator. Their action on the position and momentum operators are given by

$$\mathcal{P} : x \to -x, \quad p \to -p, \quad \mathcal{T} : x \to x, \quad p \to -p, \quad i \to -i. \tag{65}$$

For a constant mass Schrödinger Hamiltonian, the condition for $\mathcal{PT}$ symmetry reduces to $V(x) = V^*(-x)$. However in the case of position-dependent mass, an additional condition is required. To see this we note that in the present case, the Hamiltonian is of the form

$$H = -\frac{1}{2m(x)} \frac{d^2}{dx^2} - \frac{m'(x)}{2m^2(x)} \frac{d}{dx} + V(x). \tag{66}$$

From (65), it follows that the conditions for the Hamiltonian (66) to be $\mathcal{PT}$ symmetric are

$$m(x) = m(-x), \qquad V(x) = V^*(-x). \tag{67}$$

It may be pointed out that here we are working with a mass profile $m(x) = (1 + \lambda x^2)^{-1}$ which is an even function and consequently satisfies the first condition of (67). To generate non-Hermitian interaction in the present case, we introduce a complex coupling constant. As an example, let us first consider the potential appearing in (11). It can be seen from (18) that the energy for this potential does not depend on one of the potential parameters, namely B. Thus, we consider the complex potential

$$V(x) = \left[\frac{B^2 - A^2 - A\sqrt{\lambda}}{1 + \lambda x^2} + iB(2A + \sqrt{\lambda})\left(\frac{\sqrt{\lambda}x}{1 + \lambda x^2}\right) + A^2\right]. \tag{68}$$

From (68), it can be easily verified that $V(x) = V^*(-x)$ so that the Hamiltonian (66) with this potential is $\mathcal{PT}$ symmetric. In this case, the spectrum is real and given by (18). Proceeding in a similar way, we have obtained the spectrum of a number of $\mathcal{PT}$ symmetric potentials and the results are given in table 2. Incidentally, all the potentials in table 2 are shape invariant and the results can also be obtained algebraically.

Table 2. Exactly solvable $\mathcal{PT}$ symmetric potentials, where $s = \frac{A}{\sqrt{\lambda}}$, $r = i\frac{B}{\sqrt{\lambda}}$, $r_1 = i\frac{B}{\lambda}$, $a = \frac{r_1}{s-n}$, $s_1 = s-n+a$, $s_2 = s-n-a$, $s_3 = a-n-s$, $s_4 = -(s+n+a)$, $s' = \frac{A}{\sqrt{|\lambda|}}$ & $r' = i\frac{B}{\sqrt{|\lambda|}}$. The first four entries correspond to $\lambda > 0$ and the last two correspond to $\lambda < 0$.

$V(x)$	$W(x)$	E_n	$\psi_n(x)$
$\frac{-B^2-A^2-A\sqrt{\lambda}}{1+\lambda x^2} + iB(2A+\sqrt{\lambda})\frac{\sqrt{\lambda}x}{1+\lambda x^2} + A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}} + iB\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{\lambda}(2A - n\sqrt{\lambda})$	$i^n(1+\lambda x^2)^{-\frac{s}{2}}e^{-r\tan^{-1}(x\sqrt{\lambda})}$ $P_n^{(ir-s-\frac{1}{2},-ir-s-\frac{1}{2})}\left(ix\sqrt{\lambda}\right)$
$A^2 - \frac{B^2}{A^2} - \frac{A(A+\sqrt{\lambda})}{1+\lambda x^2} + i2B\frac{\sqrt{\lambda}x}{1+\lambda x^2}$, $B < A^2$	$A\frac{\sqrt{\lambda}x}{\sqrt{1+\lambda x^2}} + i\frac{B}{A}$	$A^2 - \frac{B^2}{A^2} - (A-n\sqrt{\lambda})^2 + \frac{B^2}{(A-n\sqrt{\lambda})^2}$	$\left(1 - \frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_1}{2}}\left(1 + \frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)^{\frac{s_2}{2}}$ $P_n^{(s_1,s_2)}\left(\frac{x\sqrt{\lambda}}{\sqrt{1+\lambda x^2}}\right)$
$A^2 - \frac{B^2}{A^2} - 2iB\frac{\sqrt{1+\lambda x^2}}{\sqrt{\lambda}x} + \frac{A(A-\sqrt{\lambda})}{\lambda x^2}$, $B > A^2$ $0 \leqslant x\sqrt{\lambda} \leqslant \infty$	$i\frac{B}{A} - A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}}$	$A^2 - \frac{B^2}{A^2} - (A+n\sqrt{\lambda})^2 + \frac{B^2}{(A+n\sqrt{\lambda})^2}$	$\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} - 1\right)^{\frac{s_3}{2}}\left(\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} + 1\right)^{\frac{s_4}{2}}$ $P_n^{(s_3,s_4)}\left(\frac{1+\lambda x^2}{x\sqrt{\lambda}}\right)$
$\frac{A^2-B^2+A\sqrt{\lambda}}{\lambda x^2} - iB(2A+\lambda)\frac{\sqrt{1+\lambda x^2}}{\lambda x^2}A^2$, $A < B$ $0 \leqslant x\sqrt{\lambda} \leqslant \infty$	$A\frac{\sqrt{1+\lambda x^2}}{x\sqrt{\lambda}} - iB\frac{1}{x\sqrt{\lambda}}$	$n\sqrt{\lambda}(2A - n\sqrt{\lambda})$	$(\sqrt{1+\lambda x^2}-1)^{(\frac{r-s}{2})}(\sqrt{1+\lambda x^2}+1)^{-(\frac{r+s}{2})}$ $P_n^{(r-s-\frac{1}{2},-r-s-\frac{1}{2})}(\sqrt{1+\lambda x^2})$
$\frac{A^2-B^2-A\sqrt{|\lambda|}}{1+\lambda x^2} - iB(2A-\sqrt{|\lambda|})\frac{x\sqrt{|\lambda|}}{1+\lambda x^2} - A^2$ $\frac{1}{-\sqrt{|\lambda|}} \leqslant x \leqslant \frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}} - iB\frac{1}{\sqrt{1+\lambda x^2}}$	$n\sqrt{\lambda}(2A + n\sqrt{|\lambda|})$	$(1 - x\sqrt{|\lambda|})^{(\frac{s'-r'}{2})}(1 + x\sqrt{|\lambda|})^{(\frac{r'+s'}{2})}$ $P_n^{(s'-r'-\frac{1}{2},s'+r'-\frac{1}{2})}(x\sqrt{|\lambda|})$
$\frac{A(A-\sqrt{|\lambda|})}{1+\lambda x^2} - 2iB\frac{x\sqrt{|\lambda|}}{1+\lambda x^2} - A^2 - \frac{B^2}{A^2}$ $\frac{1}{-\sqrt{|\lambda|}} \leqslant x \leqslant \frac{1}{\sqrt{|\lambda|}}$	$A\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}} - i\frac{B}{A}$	$-\frac{B^2}{A^2} - A^2 + (A+n\sqrt{|\lambda|})^2 + \frac{B^2}{(A+n\sqrt{|\lambda|})^2}$	$\left(\frac{\lambda x^2}{1+\lambda x^2} - 1\right)^{-(\frac{s'+n}{2})}e^{-a\sqrt{|\lambda|}x}$ $P_n^{(-s'-n-ia,-s'-n+ia)}\left(-i\frac{x\sqrt{|\lambda|}}{\sqrt{1+\lambda x^2}}\right)$

5. Quasi-exactly solvable $\mathcal{PT}$ symmetric potentials in PDMSE

The complex sextic potential in the constant mass Schrödinger equation has been discussed in [29]. By using transformation (12) for $\lambda > 0$, we obtain the corresponding quasi-exactly solvable potentials in PDMSE.

For $\lambda > 0$, the potential is taken as

$$V(x) = \sum_{k=1}^{6} \frac{c_k}{\lambda^{\frac{k}{2}}} (\sinh(x\sqrt{\lambda}))^{-k}, \tag{69}$$

where for $V(x)$ to be $\mathcal{PT}$ symmetric, c_1, c_3, c_5 are purely imaginary and c_2, c_4, c_6 are real.

Following [29], the ansatz for the wavefunction is taken as

$$\psi(x) = f(x) \exp\left(-\sum_{j=1}^{4} \frac{b_j}{\lambda^{\frac{j}{2}}} (\sinh(x\sqrt{\lambda}))^{-j}\right), \tag{70}$$

where $f(x)$ is some polynomial function of x. We shall focus on the following choices of $f(x)$:

(a) $f(x) = 1$

(b) $f(x) = \dfrac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} + a_0$

(c) $f(x) = \dfrac{(\sinh(x\sqrt{\lambda}))^{-2}}{\lambda} + a_1 \dfrac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} + a_0.$

For complex potentials, a_0 is purely imaginary in (b), but in (c) a_1 is purely imaginary, but a_0 is real.

Without going into the details of calculation, which are quite straightforward, let us summarize our results.

Case 1. $f(x) = 1$

In this case, the relation between the parameters c_i and b_i is found to be

$$c_1 = -3b_3 + 2b_1b_2, \quad c_2 = -6b_4 + 3b_1b_3 + 2b_2^2, \quad c_3 = 4b_1b_4 + 6b_2b_3$$
$$c_4 = 8b_2b_4 + \tfrac{9}{2}b_3^2, \quad c_5 = 12b_3b_4, \quad c_6 = 8b_4^2, \tag{71}$$

and

$$E = b_2 - \tfrac{1}{2}b_1^2. \tag{72}$$

Without loss of generality, we can choose $c_6 = \frac{1}{2}$ which fixes the leading coefficient of $V(x)$. It gives $b_4 = \pm\frac{1}{4}$. Taking the positive sign to ensure the normalizability of the wavefunction, we obtain

$$\psi(x) = \exp\Bigg(-\frac{b_1(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} - \frac{b_2(\sinh(x\sqrt{\lambda}))^{-2}}{\lambda} - \frac{b_3(\sinh(x\sqrt{\lambda}))^{-3}}{\lambda\sqrt{\lambda}}$$
$$- \frac{(\sinh(x\sqrt{\lambda}))^{-4}}{4\lambda^2}\Bigg). \tag{73}$$

Now if b_1 and b_3 are purely imaginary, then c_1, c_3, c_5 are also purely imaginary. In that case, $V(x)$ in equation (69) and $\psi(x)$ in equation (70) are $\mathcal{PT}$ symmetric and E is real.

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Case 2. $f(x) = \frac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} + a_0$, where a_0 is purely imaginary.

In this case, the wavefunction is of the form

$$\psi(x) = \left(\frac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} + a_0 \right)$$

$$\exp\left(-\frac{b_1(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} - \frac{b_2(\sinh(x\sqrt{\lambda}))^{-2}}{\lambda} - \frac{b_3(\sinh(x\sqrt{\lambda}))^{-3}}{\lambda\sqrt{\lambda}} - \frac{(\sinh(x\sqrt{\lambda}))^{-1}}{4\lambda^2} \right). \tag{74}$$

In this case, the relation between the parameters is given by

$$c_1 = -6b_3 + 2b_1b_2 + a_0, \qquad c_2 = -\tfrac{5}{2} + 3b_1b_3 + 2b_2^2, \qquad c_3 = b_1 + 6b_2b_3$$

$$c_4 = 2b_2 + \tfrac{9}{2}b_3^2, \qquad c_5 = 3b_3, \qquad c_6 = \tfrac{1}{2}. \tag{75}$$

a_0 satisfies the condition

$$a_0^3 - 3b_3a_0^2 + 2b_2a_0 - b_1 = 0. \tag{76}$$

The energy is given by

$$E = -\tfrac{1}{2}b_1^2 + 3b_2 - 3a_0b_3 + a_0^2. \tag{77}$$

We now consider two special cases.

(a) $b_1 = b_3 = 0$ and $a_0^2 < 0$.

In this case, c_1 is purely imaginary and $c_3 = c_5 = 0$. Moreover, $c_1 = a_0 = \pm i\sqrt{2b_2}$. So we get two different complex potentials corresponding to the above two values of c_1 with the same real energy eigenvalues. The potential, energy values and the eigenfunctions are given by

$$V(x) = \frac{1}{2}\frac{(\sinh(x\sqrt{\lambda}))^{-6}}{\lambda^3} + \frac{2b_2}{\lambda^2}(\sinh(x\sqrt{\lambda}))^{-4}$$

$$+ \frac{(2b_2^2 - \tfrac{5}{2})}{\lambda}(\sinh(x\sqrt{\lambda}))^{-2} \pm \frac{i\sqrt{2b_2}}{\sqrt{\lambda}}(\sinh(x\sqrt{\lambda}))^{-1}$$

$$E = b_2 > 0 \tag{78}$$

$$\psi(x) = \left(\frac{(\sinh(x\sqrt{\lambda})^{-1}}{\sqrt{\lambda}} \pm i\sqrt{2b_2} \right) \exp\left(-\frac{b_2}{\lambda}(\sinh(x\sqrt{\lambda}))^{-2} - \frac{1}{4\lambda^2}(\sinh(x\sqrt{\lambda}))^{-4} \right).$$

It can be easily seen from the above equations that the potential is $\mathcal{PT}$ symmetric, while the wavefunction is odd under $\mathcal{PT}$ symmetry.

(b) $b_1 = 0, b_3 \neq 0$

Then from (76), we get

$$a_0 = \tfrac{1}{2}\left(3b_3 \pm \sqrt{9b_3^2 - 8b_2}\right). \tag{79}$$

So in order to make a_0 imaginary, we must have $9b_3^2 - 8b_2 < 0$ or $b_3^2 = -|b_3|^2 \leqslant \tfrac{8}{9}b_2$.

In this case also there exist two different complex potentials corresponding to two values of b_3 with the same real energy eigenvalues $E = 3b_2 - 3a_0b_3 + a_0^2$.

Case 3. $f(x) = \frac{(\sinh(x\sqrt{\lambda}))^{-2}}{\lambda} + a_1\frac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} + a_0$, where a_1 is imaginary and a_0 is real.

In this case, the relation between the parameters is given by

$$a_1 = 2b_3, \qquad a_0 = \tfrac{1}{2}\left(2b_2 - b_3^2 \pm \sqrt{\left(2b_2 - 3b_3^2\right)^2 + 2}\right). \tag{80}$$

The wavefunction, energy and the potential are of the form

$$\psi_{\pm}(x) = \left[\frac{(\sinh(x\sqrt{\lambda}))^{-2}}{\lambda} + 2b_3 \frac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} + \frac{1}{2}\left(2b_2 - b_3^2 \pm \sqrt{(2b_2 - 3b_3^2)^2 + 2}\right) \right]$$

$$\exp\left(-2b_3(b_2 - b_3^2)\frac{(\sinh(x\sqrt{\lambda}))^{-1}}{\sqrt{\lambda}} - b_2 \frac{(\sinh(x\sqrt{\lambda}))^{-2}}{\lambda}\right.$$

$$\left. - b_3 \frac{(\sinh(x\sqrt{\lambda}))^{-3}}{\lambda\sqrt{\lambda}} - \frac{1}{4}\frac{(\sinh(x\sqrt{\lambda}))^{-4}}{\lambda^2}\right) \tag{81}$$

$$E_{\pm} = -2b_3^2(b_2 - b_3^2)^2 + 3b_2 - b_3^2 \pm \sqrt{(2b_2 - 3b_3^2)^2 + 2} \tag{82}$$

$$V(x) = \frac{1}{2\lambda^3}(\sinh(x\sqrt{\lambda}))^{-6} + \frac{3b_3}{\lambda^2\sqrt{\lambda}}(\sinh(x\sqrt{\lambda}))^{-5} + \frac{(2b_2 + \frac{9}{2})}{\lambda^2}(\sinh(x\sqrt{\lambda}))^{-4}$$

$$+ \frac{2b_3}{\lambda\sqrt{\lambda}}(4b_2 - b_3^2)(\sinh(x\sqrt{\lambda}))^{-3} + \frac{\left[2(b_2^2 + 3b_2b_3^2 - 3b_3^4) - \frac{7}{2}\right]}{\lambda}(\sinh(x\sqrt{\lambda}))^{-2}$$

$$+ \frac{b_3(4b_2^2 - 4b_2b_3^2 - 7)}{\sqrt{\lambda}}(\sinh(x\sqrt{\lambda}))^{-1}. \tag{83}$$

Results (80)–(83) are valid for both real and purely imaginary b_i. When b_i are purely imaginary, the potential and wavefunction are $\mathcal{PT}$ symmetric while for real b_i $\mathcal{PT}$ symmetry is broken. In particular, when b_3 is purely imaginary we have a complex $\mathcal{PT}$ symmetric two-parameter family of potentials corresponding to two values of a_0 with two distinct real eigenvalues.

6. Discussion

We have studied various exactly solvable as well as quasi-exactly solvable and non-Hermitian generalizations of the quantum nonlinear oscillator with the mass function $\left(\frac{1}{1+\lambda x^2}\right)$. We have also obtained a closed form normalization constant for the eigenfunctions of a quantum nonlinear oscillator. A relationship between the λ-deformed Hermite polynomial and Jacobi polynomial has also been found. By exploiting the supersymmetry of PDMSE, we have obtained some shape invariant potentials corresponding to this particular mass function. We have considered the shape invariance approach to PDMSE with broken supersymmetry as well. As for the future work, we feel that it would be interesting to examine the Lie algebraic symmetry of the exactly solvable potentials. In view of the fact that in the present case transformation (12) is invertible, it seems promising to study whether or not the Lie algebraic symmetry of the constant mass system can be transported back to the non-constant mass case. Another interesting area of investigation would be to study the classical analogs of some of the models (especially the $\mathcal{PT}$ symmetric ones) considered here.

Acknowledgments

Appendix

For $B = 0$, $A = \frac{\alpha}{\sqrt{\lambda}}$, the potential of equation (11) and its energy eigenvalues (18) reduce to

$$V(x) = \left(-\frac{\alpha^2}{\lambda} - \alpha\right)(1 + \lambda x^2)^{-1} + \frac{\alpha^2}{\lambda} \tag{A.1}$$

$$E_n = 2n\alpha - n\lambda. \tag{A.2}$$

For $|x| < \frac{1}{\sqrt{\lambda}}$, potential (A.1) can be written as

$$V(x) = \left(-\frac{\alpha^2}{\lambda} - \alpha\right)(1 - \lambda x^2 + \lambda^2 x^4 - \lambda^3 x^6 + \cdots) + \frac{\alpha^2}{\lambda}$$
$$= \alpha^2 x^2 - \lambda(\alpha^2 x^4 - \lambda\alpha^2 x^6 + \cdots) + \lambda(\alpha x^2 - \lambda\alpha x^4 + \cdots) - \alpha. \tag{A.3}$$

For $\lambda \to 0$, the potential reduces to

$$V(x) = \alpha^2 x^2 - \alpha. \tag{A.4}$$

It is clear from (A.4) and (A.2) that for $\lambda \to 0$, potential (11) and the energy eigenvalues (18) reduce to those of a simple harmonic oscillator.

For $A = \frac{\alpha}{\sqrt{\lambda}}$, $B = 0$ and using relation (26), the expression for the wavefunction (19) is

$$\psi_n(x) = N_n'(1 + \lambda x^2)^{-\frac{2\alpha}{\lambda}}\mathcal{H}_n\left(\sqrt{\alpha}x, \frac{\lambda}{\alpha}\right), \tag{A.5}$$

where

$$N_n' = \frac{1}{2^n n!}\left(\frac{\alpha}{\lambda}\right)^{\frac{n}{2}} N_n$$
$$= \left[\frac{\alpha^n\left(\frac{\alpha}{\lambda} - n\right)\Gamma\left(\frac{\alpha}{\lambda} - n + \frac{1}{2}\right)\Gamma\left(\frac{\alpha}{\lambda} - n + \frac{1}{2}\right)}{\pi n! 2^{2n - \frac{2\alpha}{\lambda}}\lambda^{n - \frac{1}{2}}\Gamma\left(\frac{2\alpha}{\lambda} - n + 1\right)}\right]^{1/2}. \tag{A.6}$$

Now for $\lambda \to 0$, the λ-deformed Hermite polynomial becomes the conventional Hermite polynomial H_n [21]. Consequently, at the $\lambda \to 0$ limit the unnormalized wavefunction given in equation (A.5) reduces to

$$\psi_n(x) \propto e^{-\frac{\alpha x^2}{2}} H_n(\sqrt{\alpha}x). \tag{A.7}$$

Using the asymptotic formula $\Gamma(az + b) \sim \sqrt{2\pi}\, e^{-az}(az)^{az+b-\frac{1}{2}}$ (see 6.1.39 of [24]) in (A.6), we have

$$N_n' = \left(\frac{\sqrt{\alpha} - \frac{n\lambda}{\sqrt{\alpha}}}{\sqrt{\pi}2^n n!}\right)^{1/2}. \tag{A.8}$$

Therefore from equations (A.7) and (A.8), it follows that for $\lambda \to 0$ the wavefunction given in equation (19) reduces to that of a simple harmonic oscillator.

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